The Area Model of Multiplication of Fractions

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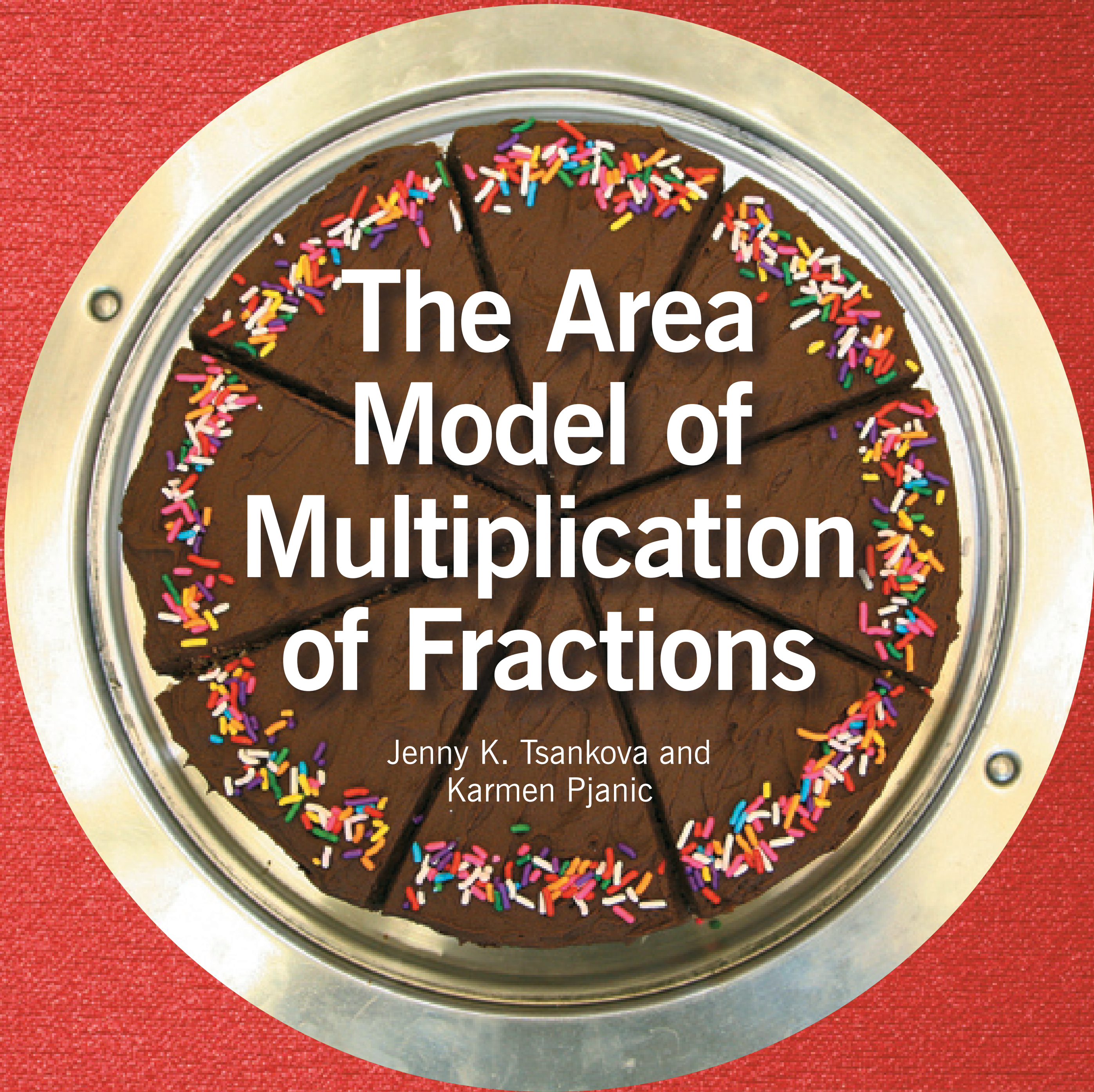

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The algorithm for multiplying proper fractions is often taught by asking students to notice patterns when finding part of a fractional part. An area-model approach will extend students' understanding.

Teaching students how to multiply fractions is challenging, not so much from a computational point of view but from a conceptual one.

The algorithm for multiplying fractions is much easier to learn than many other algorithms, such as subtraction with regrouping, long division, and certainly addition of fractions with unlike denominators. However, it has been found that students often have difficulties applying the algorithm with flexibility.

Students often do not—

1. recognize when to use the algorithm;
2. use the algorithm to multiply decimals; or
3. create an appropriate pictorial representation of a problem situation.

To develop new understanding and skills, students need to recognize how mathematical ideas connect and how they build on previously learned ideas (NCTM 2000). Teaching the concept of multiplication of fractions must be constructed on students' prior knowledge of algorithms and representations. This "knowledge package" of multiplication of fractions (Ma 1999) includes, but it is not limited to, operations with natural numbers, the meaning of *fraction* as being "part of a whole" and as "part of a set," and the concept of measurement.

A key prerequisite idea in the package is the multiplication of natural numbers. Typically, multiplication of natural numbers is introduced as repeated addition of equal groups. When used on a number line, it is a pictorial representation of an array. For example, the expression 2×4 can represent "2 baskets of 4 apples each," described numerically as 4 + 4 and illustrated as a 2×4 array. The expression 2×4 can also represent an area measurement, as in this situation:

> How many square (1 ft. × 1 ft.) tiles would you need to cover a rectangular patio that is 2 feet long and 4 feet wide?

The question is whether students' understanding of representing multiplication as repeated addition, as an array, or as area measurement will be enough knowledge when fractions are introduced or if new models will be needed. The concept of multiplication as repeated addition nicely supports multiplying a whole number by a fraction. It is clear to students that $3 \times 1/2$ can be represented numerically, pictorially, or on a number line as 1/2 + 1/2 + 1/2. However, the model of repeated addition is not appropriate to represent multiplication of two proper fractions.

IT'S A PIECE OF CAKE

Multiplying two fractions is usually introduced with the idea of finding a fraction of a fractional part, as described in this problem:

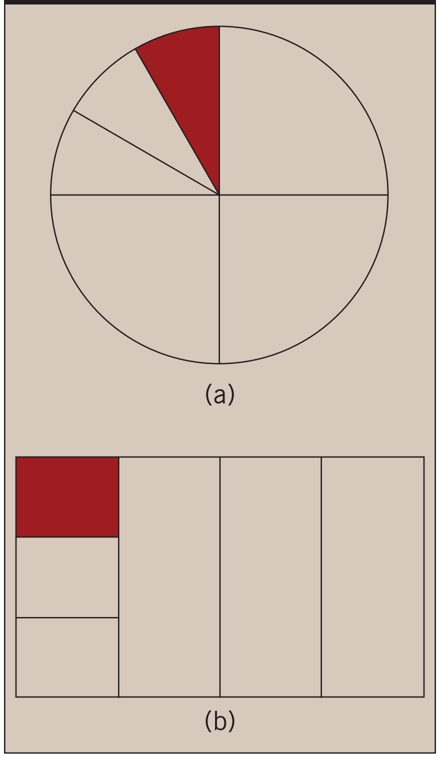
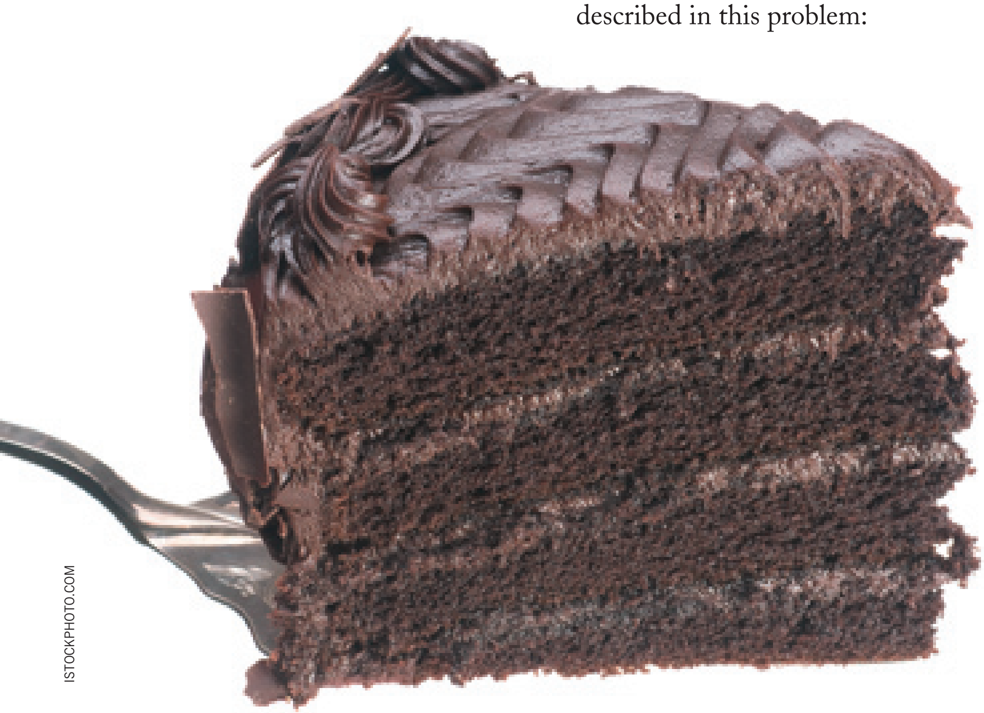


Fig. 1 A circular and a rectangular diagram showing 1/3 of 1/4 can fix the image in students' minds before they explore an algorithm for multiplying fractions.

> Melinda had a birthday yesterday. A quarter (1/4) of the birthday cake remained. This morning, Melinda ate 1/3 of the remaining quarter cake. What fraction of the whole cake did Melinda eat this morning?

The concept of finding a fraction of a fractional part is new to students. It is challenging for them to design a mathematical model of writing an expression using a new concept. **Figure 1** shows both a circular and rectangular model of 1/3 of 1/4. Students can make sense out of a problem when they can see it represented in various ways:

- By folding paper that shows a whole divided into four equal pieces, with each quarter divided into three equal pieces
- By drawing and dividing a segment on a number line



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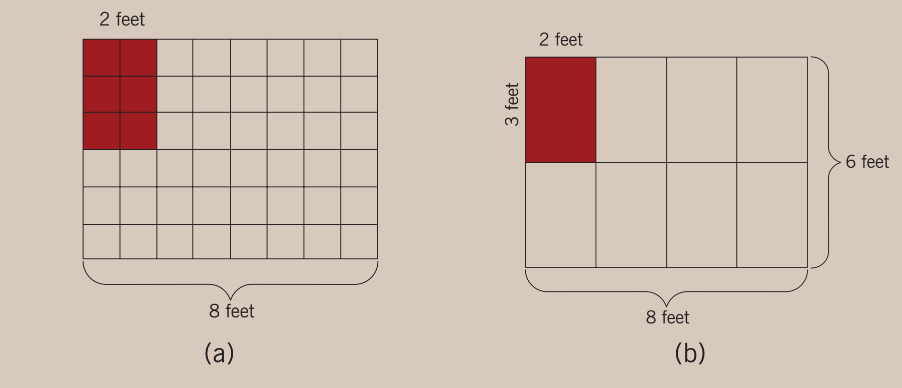
- By representing the part of a part as a double-shaded section

These representations can help build students' understanding when they must solve a problem that implies taking part of a part. Later, the algorithm for multiplying fractions is habitually derived by observing a pattern that emerges from numerous problem situations in which students find part of a fractional part.

Unfortunately, these models fail to relate the concept of multiplication of natural numbers to the concept of multiplication of fractions. Natural number multiplication is encountered by students as a measurement concept of finding the area of a rectangle. Multiplying with fractions uses problem situations in which fractions are multiplied to obtain the area of a shape. It is obvious to us that a different model or another model is needed to connect these concepts. The area model does just that.

Fig. 2 Starting with a 6 ft. × 8 ft. quilt, finding the fractional part of a 2 ft. × 3 ft. panel can be represented in at least two ways, as shown.



SEWING UP A NEW MODEL

Consider the following problem and some possible pictorial representations in **figure 2**:

Betsy's mom is sewing a quilt that is 8 feet long and 6 feet wide. Betsy wants to work on the quilt, too. Her mom asked her to sew a rectangular piece of the quilt that is 2 feet in length and 3 feet in width. What fraction of the area of the whole rectangular quilt will Betsy sew?

The concept of multiplication applied in finding the area of a rectangle connects with the prior understanding that students have about multiplying natural numbers.

Possible explanations of solutions for **figure 2a** and **figure 2b** follow:

1. Each square represents 1 ft. × 1 ft. The shaded rectangle is Betsy's part of the quilt, which is 3 feet long and 2 feet wide. The area of Betsy's rectangular quilt is 2 ft. × 3 ft. = 6 sq. ft. The area of Betsy's quilt is 6 out of 48 equal pieces, or 6/48, or 1/8, of the area of the whole quilt.
2. The area of Betsy's rectangular quilt is 2 ft. × 3 ft. = 6 sq. ft., although the whole is not divided into 1 ft. × 1 ft. squares. The area of Betsy's quilt is now 1 out of 8, or 1/8, of the area of the whole quilt.

This problem solution involves finding the area of a rectangle and considering it to be a fractional part of the whole area. To facilitate students' reasoning, review some key mathematical ideas and build on them to extend students' understanding. For instance, when the product of natural numbers is represented with an array (e.g., 3 × 2, or 3 groups of 2), it is important to point out that one factor is placed horizontally and the other factor is placed vertically in the array.

Similarly, if we are to find the area of a rectangle with dimensions 3 units and 2 units, the length of the rectangle can be represented as 3 units horizontally and the width can be represented vertically as 2 units (or vice versa). The area of the rectangle will be the product 3 × 2, or 6, square units (1 × 1).

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Students must recognize two key ideas. First, they should see that a multiplicative expression, $a \times b$, can be represented by a measurement model of finding the area of rectangle, or $l \times w$. Second, students should connect in the other direction that a measurement model of area of a rectangle leads to a multiplicative expression $a \times b$. Thus, the concept of multiplication applied in finding the area of a rectangle connects with the prior understanding that students have about multiplying natural numbers. It also supports their work with multiplication of fractions.

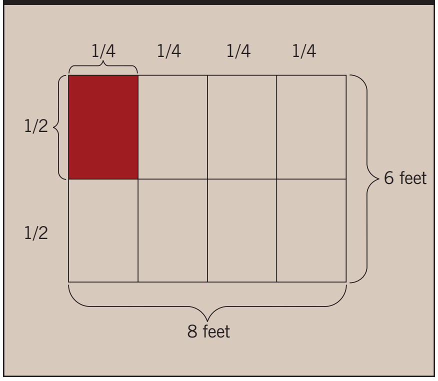
Observe the slight change in the quilt-sewing problem:

> Betsy's mom is sewing a quilt that is 8 feet long and 6 feet wide. Betsy wants to work on the quilt, too. Her mom asked her to sew a rectangular piece of the quilt with length 1/4 of the length of the quilt and width 1/2 of the width of the quilt. What fraction of the area of the whole rectangular quilt will Betsy sew?

In this problem, the length and the width measurements of the rectangle are fractional parts of the whole quilt, so students can easily visualize that the obtained rectangle will be smaller than the original rectangular quilt. Proceeding logically from the established connection between multiplication and measurement representation, one can see that the measurements of the new rectangle needed are 1/4 the length (shown horizontally; see **fig. 3**), and 1/2 the width (shown vertically) of the dimensions of the rectangular quilt. Thus, one needs to split the length into four equal parts and the width into two equal parts. The smaller rectangle that is needed is shaded in **figure 3**.

One way to think about the area of the shaded rectangle is to determine both 1/4 of 8 feet and 1/2 of 6, multiply the whole numbers, and finish the problem as with the previous question. Another way to think about this problem is to recognize that the area of the shaded rectangle is the product of 1/4 × 1/2. When assisted by the diagram and guided by the teacher, students will notice that Betsy's rectangle is some fractional part of the whole quilt. Since fractional pieces must be congruent, completing the division of the whole into equal parts will naturally lead students to the answer of Betsy's quilt piece being 1/8 of the whole.

Fig. 3 Starting with a 6 ft. × 8 ft. quilt, finding the dimensions from a fractional part that is 1/4 of the total width and 1/2 of the total length can be represented as shown.



HOW THE AREA MODEL OF MULTIPLICATION SHAPES UP

This model could truly be referred to as *the area model of multiplication of fractions* because it builds on prior knowledge of multiplication of natural numbers as finding the area of a rectangle. It allows students to generalize the algorithm of multiplication of proper fractions, in which the product is less than each of the factors, which is traditionally a challenging and counterintuitive idea for students. The area model also naturally leads to the distributive property of multiplication in general and even to the multiplication of binomials.

Mathematical ideas should not be taught in isolation. The concept of multiplication of fractions and its representation should not be an exception. Developing and understanding the algorithm for multiplying proper fractions should be placed on the foundation of understanding the algorithm for multiplication of natural numbers and its connection to the measurement concept of finding the area of a rectangle. Thus, it is important for students in upper elementary school and middle school to have that groundwork laid and the connections among concepts explicitly made.

If we want our students to make sense of concepts and apply skills with mastery and flexibility that stand the test of time, we need to make sense of how we establish connections.

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